SELF-SIMILAR SOLUTION OF THE PROBLEM OF THREE-DIMENSIONAL SPREADING OF A NONLINEARLY-VISCOUS FLUID

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It is well-known [1, 2] that the study of hydrodynamic processes in the boundary layer approximation can be reduced to the solution of an equation describing the shape of the free surface of a nonlinearly-viscous fluid using the methods of similarity theory and dimensional analysis. However, the boundary value problems arising in connection with the mathematical modelling are essentially nonlinear and their solution, in the general case, can only be found by numerical methods. The absence of a priori estimates of the accuracy of the numerical methods makes it necessary to construct analytical solutions of these problems at least in particular self-similar cases for testing appropriate difference schemes on them.

In [3], using the notion of group analysis [4, 5], a group classification was made of the equation describing the shape of the free surface of a nonlinearly-viscous fluid in a one-dimensional approximation and, in particular, the shape of a glacier. The invariant solutions obtained were used to construct self-similar problems illustrating qualitative features of the flow of the fluid with a rheological power law.

In the present paper we solve a two-dimensional self-similar problem concerned with the spread of a fluid over a bed of complex configuration.

1. The Basic Equation. Considering the nonstationary flow of a nonlinearly-viscous fluid in an isothermal approximation, we can show [2] that the function $\ell(x, y, t)$ describing the free surface satisfies a second-order nonlinear partial differential equation, namely,

$$\frac{\partial l}{\partial t}=\frac{\partial}{\partial x}\left\{\left[\frac{\partial l}{\partial x}\Big/\sqrt{\left(\frac{\partial l}{\partial x}\right)^2+\left(\frac{\partial l}{\partial y}\right)^2}\right]\int\limits_{l_0}^{l}(l-z)\,\Gamma\left[(l-z)\sqrt{\left(\frac{\partial l}{\partial x}\right)^2+\left(\frac{\partial l}{\partial y}\right)^2}\right]dz\right\}+$$
$$+\frac{\partial}{\partial y}\left\{\left[\frac{\partial l}{\partial y}\Big/\sqrt{\left(\frac{\partial l}{\partial x}\right)^2+\left(\frac{\partial l}{\partial y}\right)^2}\right]\int\limits_{l_0}^{l}(l-z)\,\Gamma\left[(l-z)\sqrt{\left(\frac{\partial l}{\partial x}\right)^2+\left(\frac{\partial l}{\partial y}\right)^2}\right]dz\right\}, \quad (1)$$

where t is the time; x and y are spatial coordinates; $\ell_0(x, y)$ is the profile of the surface over which spreading of the fluid takes place; $\Gamma(z) = kz^{\alpha}$ is a function characterizing rheological properties of the fluid with a power-law dependence.

From Eq. (1), with appropriate boundary conditions on $\ell(x, y, t)$, we can determine all the other characteristics of the flow; in particular, the velocity in an arbitrary direction, the stress arising in the fluid, and so forth.

2. Invariant Solutions. Construction of a full spectrum of invariant solutions of a specific differential equation is based on its group properties [4, 5]. Therefore, with the aim of obtaining invariant solutions of Eq. (1) we carried out a group classification for it. Detailed results of the group analysis are given in [3]. Here we give invariant solutions needed in the sequel only of the second rank (Table 1).

3. Solution of the Problem of Three-dimensional Spreading of the Fluid. Among the invariant solutions, a most general and nontrivial solution is $\langle\lambda X_4 + X_5\rangle$, which depends on the arbitrary parameters α and λ. As can be seen from Table 1, this solution can be found in the form $I_3 = \psi(I_1, I_2)$ or

$$u=t^{\frac{\alpha+1-\lambda}{\lambda(2\alpha+1)}}\psi(I_1,I_2),\ I_1=\frac{x}{y},\ I_2=\frac{y}{t^{1/\lambda}} \quad (2)$$

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($u = \ell - \ell_0$ is the thickness of the layer of nonlinearly-viscous fluid; ψ is the sought-for function, depending on invariants I_1 and I_2).

With the aid of the invariant solution (2) we obtain an exact solution of the problem of the three-dimensional spreading of the fluid, concentrated initially at the coordinate origin O(0, 0), over the channel $\ell_0 = (x^2/y^2 + C)/y^2$ (C = const). In Eq. (2) we take $\lambda = 5\alpha + 3$ and append to the right side of Eq. (1) the function

$$F = t^{-\frac{5(\alpha+1)}{5\alpha+3}} f_0(I_1, I_2) \tag{3}$$

[$f_0(I_1, I_2)$ is an arbitrary function].

To complete the system of equations we write the initial and boundary conditions

$$t = 0\colon \Omega_1(t) \to 0; \tag{4}$$

$$r \to 0\colon\ l = l_0 + t^{-\frac{2}{5\alpha+3}} \Phi(t, \varphi); \tag{5}$$

$$(l - l_0)|_\Gamma = 0; \tag{6}$$

$$Q|_\Gamma = Q_0 = 0, \tag{7}$$

where $r = \sqrt{x^2 + y^2}$; $\varphi = \operatorname{arctg}(x/y)$; $\Phi(t, \varphi)$ is an arbitrary assigned function; Q is the fluid mass flow; $\Omega_1(t)$ is the region of computation; Γ is the region boundary. Substituting Eqs. (2) and (3) into Eq. (1), we obtain a differential equation for determination of the function $\psi(I_1, I_2)$ with one less independent variable:

$$\begin{aligned}
&\frac{\partial}{\partial I_1}\left\{\psi^{\alpha+2}\left[\left(\frac{1}{I_2}\frac{\partial\psi}{\partial I_1} + \frac{2I_1}{I_2^3}\right)^2 + \left(\frac{\partial\psi}{\partial I_2} - \frac{I_1}{I_2}\frac{\partial\psi}{\partial I_1} - \frac{2\left(C + 2I_1^2\right)}{I_2^3}\right)^2\right]^{\frac{\alpha-1}{2}} \times\right.\\
&\left.\times\left[\left(\frac{1}{I_2}\frac{\partial\psi}{\partial I_1} + \frac{2I_1}{I_2^3}\right) - I_1\left(\frac{\partial\psi}{\partial I_2} - \frac{I_1}{I_2}\frac{\partial\psi}{\partial I_1} - \frac{2\left(C + 2I_1^2\right)}{I_2^3}\right)\right]\right\} +\\
&+\frac{\partial}{\partial I_2}\left\{\psi^{\alpha+2}\left[\left(\frac{1}{I_2}\frac{\partial\psi}{\partial I_1} + \frac{2I_1}{I_2^3}\right)^2 + \left(\frac{\partial\psi}{\partial I_2} - \frac{I_1}{I_2}\frac{\partial\psi}{\partial I_1} - \frac{2\left(C + 2I_1^2\right)}{I_2^3}\right)^2\right]^{\frac{\alpha-1}{2}} \times\right.\\
&\left.\times\left[I_2\left(\frac{\partial\psi}{\partial I_2} - \frac{I_1}{I_2}\frac{\partial\psi}{\partial I_1} - \frac{2\left(C + 2I_1^2\right)}{I_2^3}\right)\right] + \frac{I_2\psi}{3 + 5\alpha}\right\} = -I_2 f_0.
\end{aligned} \tag{8}$$

Thus, in solving problems concerned with spreading of a fluid it is necessary to specify the computational domain boundary Γ and the two boundary conditions (6) and (7). However, by virtue of the fact that Eq. (8) is nonlinear and boundary Γ is not known, we adopt a different approach for solving problem (1), (3)-(7). We specify in invariant variables the computational domain $\Omega_2 = \Omega + \Gamma_1 + \Gamma_2$ (Fig. 1), where Ω is the interior part of the semi-circle $I_1{}^2 + I_2{}^2 = R^2$; Γ_1, Γ_2 are boundaries of the domain. Then, having the computational domain, it is sufficient to indicate only one condition on Γ_2. In the I_1 and I_2 variables this condition, and also the condition (5) at zero, assume the form

$$\psi|_{\Gamma_1} = \psi_0(I_1), \quad \psi|_{\Gamma_2} = 0. \tag{9}$$

Proceeding, we find the solution of problem (8), (9). According to this solution, we recover the form of the solution for problem (1), (3)-(7) in the domain $\Omega_1 = \Omega + \Gamma + O(0, 0)$ (Fig. 2), in which Ω_2 appears, and we determine the mass flow of the fluid on boundary Γ in the following way:

$$q_x = \left\{\frac{\partial l}{\partial x}(l - l_0)^{\alpha+2}\left[\sqrt{\left(\frac{\partial l}{\partial x}\right)^2 + \left(\frac{\partial l}{\partial y}\right)^2}\right]^{\alpha-1}\right\}_{|\Gamma},$$

$$q_y = \left\{\frac{\partial l}{\partial y}(l - l_0)^{\alpha+2}\left[\sqrt{\left(\frac{\partial l}{\partial x}\right)^2 + \left(\frac{\partial l}{\partial y}\right)^2}\right]^{\alpha-1}\right\}_{|\Gamma}$$

(q_x is the mass flow of the fluid in the direction of the OX axis; q_y is the mass flow of the fluid in the direction of the OY axis). This is one of the possible ways of solving the model problem (1), (3)-(7).

TABLE 1

Optimal one-parameter subalgebras	Invariants	Invariant solutions of rank II	Shape of channel
$\langle \lambda X_4 + X_5 \rangle$ $\forall\lambda$	$I_1 = \frac{x}{y}$, $I_2 = \frac{y^\lambda}{t}$, $I_3 = \frac{t}{u^{\frac{\lambda(2\alpha+1)}{\alpha+1-\lambda}}}$, $I_4 = \frac{u^{\frac{1-2\lambda}{\alpha+1-\lambda}}}{v}$, $I_5 = \frac{v}{w}$	a) $u = t^{\frac{\alpha+1-\lambda}{\lambda(2\alpha+1)}} \psi(I_1, I_2)$ b) $u = t^{\frac{\alpha+1-\lambda}{\lambda(2\alpha+1)}} \psi\left(\frac{r^\lambda}{t}\right)$	a) $l_0 = C + y^{\frac{\alpha+1-\lambda}{2\alpha+1}} f(I_1)$ b) $l_0 = Cr^{\frac{\alpha+1-\lambda}{2\alpha+1}}$, $C = \text{const}$
$\langle \sin\beta X_1 + \cos\beta X_2 \rangle$ $\forall\beta$	$I_1 = x\cos\beta - y\sin\beta$, $I_2 = t$, $I_3 = u$, $I_4 = v$, $I_5 = w$	$u = \psi(I_1, I_2)$	$l_0 = \text{const}$
$\langle X_3 \rangle$	$I_1 = x$, $I_2 = y$, $I_3 = u$, $I_4 = v$, $I_5 = w$	$u = \psi(I_1, I_2)$	l_0 — arbitrary
$\langle \sin\beta X_1 + \cos\beta X_2 + \gamma X_4 \rangle$ $\gamma \neq 0$	$I_1 = x\cos\beta - y\sin\beta$, $I_2 = \gamma y - \cos\beta \ln t$, $I_3 = tu^{2\alpha+1}$, $I_4 = \frac{u^2}{v}$, $I_5 = \frac{v}{w}$	$u = t^{-\frac{1}{2\alpha+1}} \psi(I_1, I_2)$	$l_0 = \text{const}$

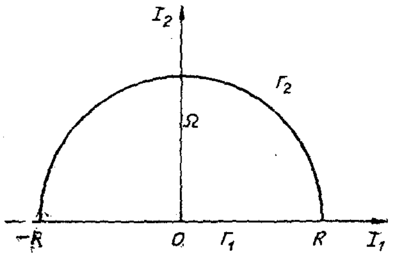

Fig. 1

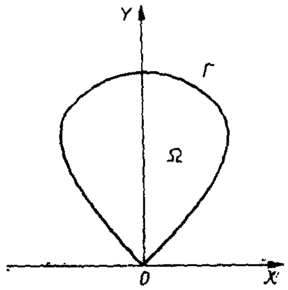

Fig. 2

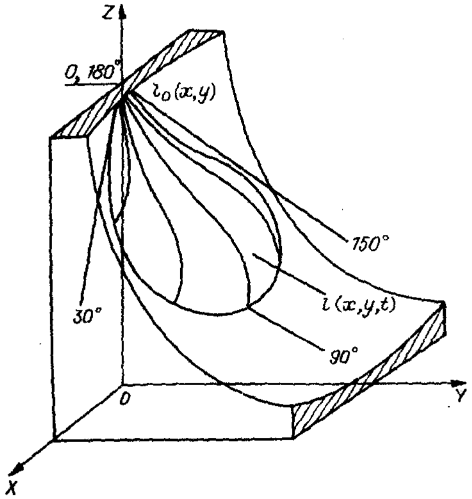

Fig. 3

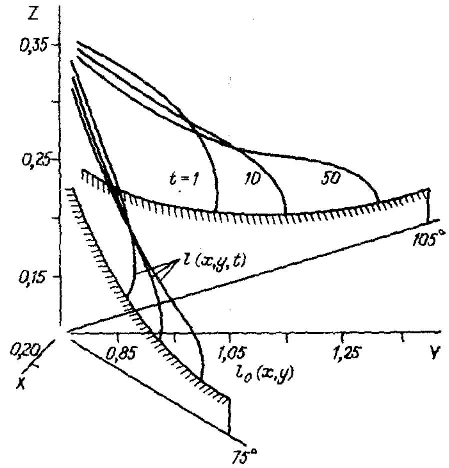

Fig. 4

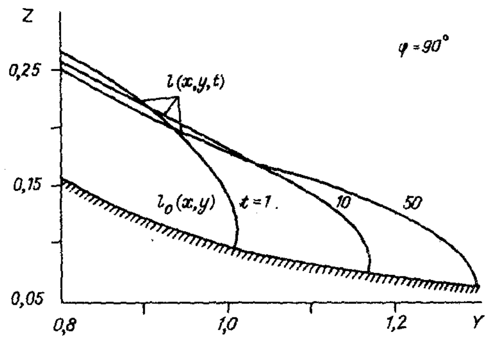

Fig. 5

TABLE 2

r	φ°					
	15, 175	30, 150	45, 135	60, 120	75, 105	90
	$\psi(r, \varphi)$					
0	1,2732	1,2732	1,2732	1,2732	1,2732	1,2732
0,1	1,0957	1,0953	1,0951	1,0951	1,0953	1,0955
0,2	0,7744	0,7744	0,7744	0,7721	0,7735	0,7744
0,3	0,5227	0,5211	0,5195	0,5192	0,5202	0,5209
0,4	0,3591	0,3576	0,3564	0,3563	0,3569	0,3573
0,5	0,2554	0,2544	0,2538	0,2539	0,2543	0,2545
0,6	0,1885	0,1879	0,1877	0,1878	0,1881	0,1882
0,7	0,1439	0,1436	0,1435	0,1437	0,1438	0,1438
0,8	0,1131	0,1128	0,1128	0,1131	0,1132	0,1132
0,9	0,0915	0,0910	0,0910	0,0924	0,0954	0,0972
1,0	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000

4. Difference Scheme and Numerical Solution. Because Eq. (8) is nonlinear we solve problem (8), (9) numerically using the method given in [6]. A difference scheme of second-order accuracy was constructed based on an integrating identity. To do this numerically we applied a two-step iterational process with a Laplace difference operator on the upper layer. Inversion of the Laplace operator was effect by the method of variable directions.

In solving problem (8), (9) we used, as initial data for function ψ, the interior of domain Ω and on boundary Γ_1 we used an analog of the Dirac delta-function in the form

$$\psi(r, \varphi) = \beta/\pi(\beta^2 r^2 + 1),\ \beta = 4,\ \forall\varphi,\ \alpha = 2.5.$$

After 200 iterations we obtained the solution shown in Table 2. The relative error amounted to 0.03%.

The general form of the two-dimensional spreading of the fluid over the channel $\ell_0 = (x^2/y^2 + 0.1)/y^2$, as well as a profile of the surface at 75°, 90°, and 105°, is shown in Figs. 3-5. Moreover, in calculating the mass flow of the fluid on boundary Γ according to the solution obtained, it turned out that the flow is equal to zero to within five-figure accuracy, which testified to our good choice of the computational domain Ω_2 for the problem (1), (3)-(7) posed here.

The form of the surface of the fluid and the coordinates at an arbitary moment of time may be determined from Eqs. (2):

$$x = yr\cos\varphi,\ r \in [0, 1],\ \varphi \in [0, \pi],$$
$$y = t^{\frac{1}{5\alpha+3}} r \sin\varphi,\quad l = l_0 + t^{-\frac{2}{5\alpha+3}}\psi(r, \varphi) \qquad (10)$$

(r, φ, ψ are shown in Table 2). From the form of solution (10) and Figs. 4 and 5 we see that over time the edge of the nonlinearly-viscous fluid advances forward and the thickness decreases, i.e., a spatial spreading is observed.

The self-similar solution obtained expresses qualitative regularities of the flow and can serve as a basis for developing numerical methods for solving problems involving substantial three-dimensional spreading of a nonlinearly-viscous fluid.

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